On McKinsey-Tarski Theorem

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Outline

Logical Preliminaries

The Basic Modal Language: Syntax

We first recall what is the basic modal language.

'Alphabets' or the set of symbols:

- propositional variables: $p, q, r, \ldots,$
- logical symbols: $\bot, \wedge, \neg,$
- modal operator: $\Diamond,$
- parantheses: $(,).$

$$\mathcal{S} = \text{Set of symbols} = \{p, q, r, \ldots, \bot, \wedge, \neg, \Diamond, (,)\}$$

The Basic Modal Language: Syntax (Cont'd)

Definition (Formulas)

Formulas are finite sequences on the set of symbols such that

1. every propositional variable is a formula,
2. $\bot$ is a formula,
3. if φ is a formula, then $\neg\varphi$ is a formula,
4. if both φ and ψ are formulas, then $(\varphi \wedge \psi)$ is a formula,
5. if φ is a formula, then $\Diamond\varphi$ is a formula, and
6. nothing else is a formula.

Examples

Some formulas: p, $\neg r$, $\Diamond\neg\bot$, $\neg(\bot \wedge (\Diamond p \wedge r))$, $\neg\neg\Diamond\neg q$.
Some strings which are not formulas: $\bot\neg$, $pq \wedge \neg), \neg\neg\bot \wedge$.

Common Abbreviations

Abbreviations

- $(\varphi \vee \psi) := \neg(\neg\varphi \wedge \neg\psi)$,
- $(\varphi \to \psi) := (\neg\varphi \vee \psi)$,
- $(\varphi \leftrightarrow \psi) := ((\varphi \to \psi) \wedge (\psi \to \varphi))$,
- $\top := \neg\bot$,
- $\Box\varphi := \neg\Diamond\neg\varphi$.

Writing parentheses is skipped, if the context is clear. For example, we may write $p \to \Box q$ instead of $(p \to \Box q)$.

Kripke Semantics

Definition (Frames)

A frame for the basic modal language is a pair $\mathfrak{F} = (W, R)$, where

1. W is a non-empty set,
2. R is a binary relation on W.

Elements of W are also called the states of W.

Let Φ denote the set of propositional variables, i.e.

$$\Phi = \{p, q, r, \ldots\}.$$

Definition (Models)

A model $\mathfrak{M}$ is a tuple $(\mathfrak{F}, V)$, where

1. $\mathfrak{F} = (W, R)$ is a frame,
2. V is a function from Φ to the powerset of W (denoted by $\mathcal{P}(W)$).

For a model $\mathfrak{M} = (\mathfrak{F}, V)$, $\mathfrak{F}$ is called the underlying frame and V is said to be a valuation on $\mathfrak{F}$.

Truth and Satisfiability

Definition (Truth)

Let w be a state in a model $\mathfrak{M} = (W, R, V)$. Then we inductively define the notion of a formula φ being satisfied (or true) in $\mathfrak{M}$ at a state w as follows:

1. $\mathfrak{M}, w \vDash p$ iff $w \in V(p)$, where $p \in \Phi$,
2. $\mathfrak{M}, w \vDash \bot$ never,
3. $\mathfrak{M}, w \vDash \neg\varphi$ iff it's not the case that $\mathfrak{M}, w \vDash \varphi$ (denoted by $\mathfrak{M}, w \nvDash \varphi$),
4. $\mathfrak{M}, w \vDash (\varphi \wedge \psi)$ iff both $\mathfrak{M}, w \vDash \varphi$ and $\mathfrak{M}, w \vDash \psi$ hold, and
5. $\mathfrak{M}, w \vDash \Diamond\varphi$ iff there exists a $v \in W$ such that Rwv and $\mathfrak{M}, v \vDash \varphi$.

A Topological Interpretation

We will use the basic modal language to describe topological spaces.[1]

Definition (Topo-models)

A topo-model is a 3-tuple (X, τ, v), where (X, τ) is a topological space and v is a function from Φ to $\mathcal{P}(X)$. Here v is said to be a valuation on X.

[1] Aiello, Pratt-Hartmann, van Bentham: Handbook of Spatial Logics (2007)

Topo-models: An Example

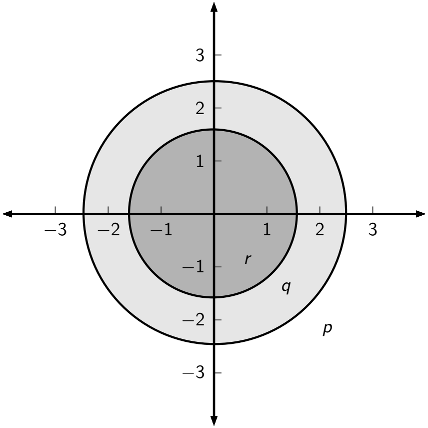

Figure: A topo-model based on $\mathbb{R}^2$

A Topological Interpretation

Definition (Basic Topological Semantics)

Truth of modal formulas is defined inductively at points x of X in a topo-model $M = (X, \tau, v)$:

1. $M, x \vDash p$ iff $x \in v(p)$, for each proposition variable p,
2. $M, x \vDash \neg\varphi$ iff it's not the case that $M, x \vDash \varphi$,
3. $M, x \vDash (\varphi \wedge \psi)$ iff both $M, x \vDash \varphi$ and $M, x \vDash \psi$ hold,
4. $M, x \vDash \Diamond\varphi$ iff for each $U \in \tau$ containing x, there exists a $y \in U$ such that $M, y \vDash \varphi$.

Remark

For any point x, if $M, x \vDash \varphi$, then $M, x \vDash \Diamond\varphi$.

An Example

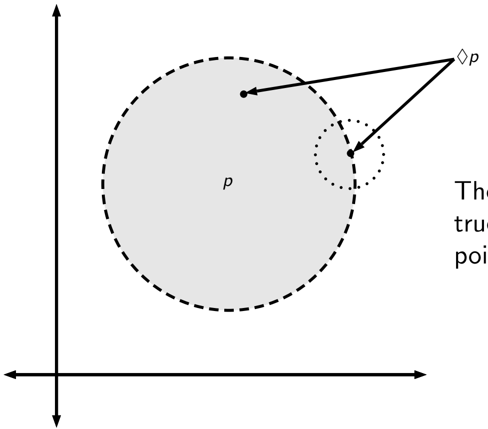

Figure: A topo-model based on $\mathbb{R}^2$

The set of all points where $\Diamond p$ is true is the closure of the set of all points where p is true.

$\Diamond$ as the Closure

Let $M = (X, \tau, v)$ be a topomodel. For a formula φ, let $[[\varphi]]$ denote all the points at which φ is true, i.e.

$$[[\varphi]] = \{x \in X \mid M, x \vDash \varphi\}.$$

Then, $y \in [[\Diamond\varphi]]$

$\Leftrightarrow$ for each $U \in \tau$ containing y, there exists some $z \in U$ such that $M, z \vDash \varphi$

$\Leftrightarrow$ for each $U \in \tau$ containing y, there exists some $z \in U$ such that $z \in [[\varphi]]$

$\Leftrightarrow$ for each $U \in \tau$ containing y, $U \cap [[\varphi]] \neq \emptyset$

$\Leftrightarrow$ $y \in$ Closure of $[[\varphi]]$.

Unravelling the Abbreviations

It can be checked that

- $M, x \vDash (\varphi \vee \psi)$ iff $M, x \vDash \varphi$ holds or $M, x \vDash \psi$ holds,
- $M, x \vDash (\varphi \rightarrow \psi)$ iff if $M, x \vDash \varphi$ holds, then $M, x \vDash \psi$ holds, and
- $M, x \vDash (\varphi \leftrightarrow \psi)$ iff either both $M, x \vDash \varphi$ and $M, x \vDash \psi$ hold, or both $M, x \nvDash \varphi$ and $M, x \nvDash \psi$ hold.

Unravelling the Abbreviations (Cont'd)

Also, $M, x \models \Box\varphi$

$\Leftrightarrow M, x \models \neg\Diamond\neg\varphi$

$\Leftrightarrow M, x \not\models \Diamond\neg\varphi$

$\Leftrightarrow$ it's not the case that for each $U \in \tau$ containing x, there exists a $y \in U$ such that $M, y \models \neg\varphi$

$\Leftrightarrow$ there exists some $U_0 \in \tau$ containing x, such that for each $z \in U_0$, we have $M, z \not\models \neg\varphi$

$\Leftrightarrow$ there exists some $U_0 \in \tau$ containing x, such that for each $z \in U_0$, we have $M, z \models \varphi$.

$\Box$ as the Interior

It can be checked that for a formula φ, we have

$$[[\Box\varphi]] = \text{ Interior of } [[\varphi]].$$

Also, we have the following:

$$[[\neg\varphi]] = [[\varphi]]^c$$
$$[[\varphi \wedge \psi]] = [[\varphi]] \cap [[\psi]]$$
$$[[\varphi \vee \psi]] = [[\varphi]] \cup [[\psi]]$$

Talking about spaces: An Example

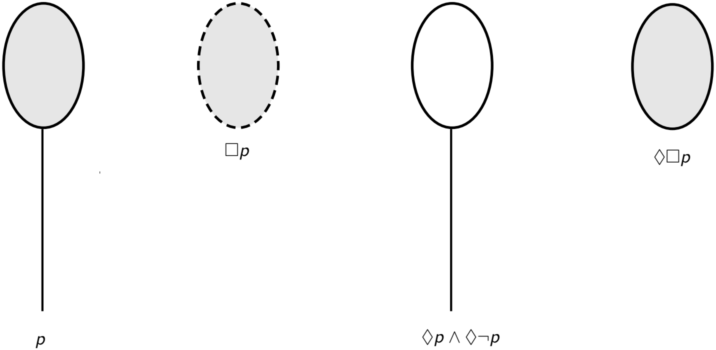

Figure: A spoon in $\mathbb{R}^2$

Validity

Definition (Validity)

A formula φ is valid on a topological space (X, τ) if φ is true at every point on every topo-model based on (X, τ) (notation: $(X, \tau) \vDash \varphi$).

A formula φ is valid on a class of topological spaces S if φ is valid on every member of S.

Validity: An Example

Example

The formula (Dual) given by

$$\Diamond p \leftrightarrow \neg \Box \neg p$$

which is just the abbreviation of

$$\Diamond p \leftrightarrow \neg\neg \Diamond \neg\neg p$$

is valid on the class of topological spaces,
as, for any topo-model,

- $M, x \vDash \Diamond p$ iff $x \in [[\Diamond p]]$ iff $x \in \mathrm{Cl}([[p]])$,
- $M, x \vDash \neg\neg\Diamond\neg\neg p$ iff $x \in [[\neg\neg\Diamond\neg\neg p]]$ iff $x \in \mathrm{Cl}([[p]]^{c\ c})^{c\ c}$.

Propositional Tautologies[2]

Propositional formulas are modal formulas which don't have an occurence of $\Diamond$ (or $\Box$).
Propositional tautologies are propositional formulas which are valid on every frame.

Remark

Propositional tautologies actually are formulas which are 'tautologies' (always true under any interpretation) in a language called Sentential Language.

The Basic Modal Language is an extension of the Sentential Language.

Examples

The formulas $p \vee \neg p$, $p \leftrightarrow \neg\neg p$, $(p \to q) \leftrightarrow (\neg q \to \neg p))$ are all examples of propositional tautologies.

[2] Enderton: A Mathematical Introduction to Logic (2001)

Normal Modal Logics

Definition (Normal Modal Logics)

A normal modal logic (or normal logic) Λ is a set of modal formulas that contains:

- all propositional tautologies,
- (K) $\Box(p \to q) \to (\Box p \to \Box q)$ and
- (Dual) $\Diamond p \leftrightarrow \neg\Box\neg p$,

and is closed under

- modus ponens (i.e., if $\varphi \in \Lambda$ and $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$),
- uniform substitution (i.e., if φ belongs to Λ, then so do all of its substitution instances), and
- generalization (i.e., if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$).

If $\varphi \in \Lambda$, we say φ is a theorem of Λ (notation: $\vdash_\Lambda \varphi$).

Normal Modal Logics: Examples

Examples

- The set of all modal formulas is a normal logic.
- If F is a class of frames, then the set of formulas valid on each element of F is a normal logic.

It can be proved that if $\{\Lambda_i \mid i \in I\}$ is a collection of normal logics, then $\cap_{i \in I} \Lambda_i$ is also a normal logic.

For a collection of modal formulas Γ, the smallest normal logic containing Γ is denoted by $\mathbf{K}\Gamma$, which is the intersection of all normal logics which contain Γ.

The Bottom Up Approach

Consider the following construction:

- $C_0 := \{\text{Propositional tautologies}\} \cup \{(\text{K})\} \cup \{(\text{Dual})\} \cup \Gamma$.
- For each $n \in \mathbb{N}$, $C_n := C_{n-1} \cup \{$all modal formulas that can be obtained by applying the rules of modus ponens, uniform substitution and generalization on $C_{n-1}\}$.

It can be proved that

$$\mathbf{K}\Gamma = \bigcup_0^\infty C_n.$$

Thus, the theorems of $\mathbf{K}\Gamma$ are exactly the formulas which can be obtained from C_0 by applying the rules a finite number of times.

The logic **S4**

S4 has been defined to be the smallest normal logic containing the following axioms:

$$\begin{array}{ll} (\mathrm{T}) & p \rightarrow \Diamond p, \\ (4) & \Diamond\Diamond p \rightarrow \Diamond p. \end{array}$$

Topological Soundess and Completeness

Definition (Topological Soundness)

A normal logic Λ is said to be sound with respect to a class of topological spaces S, if every theorem of Λ is valid on S, i.e.

$$\vdash_{\Lambda} \varphi \Rightarrow \mathsf{S} \models \varphi.$$

Definition (Topological Completeness)

A normal logic Λ is said to be complete with respect to a class of topological spaces S, if every formula that is valid on S, is theorem of Λ, i.e.

$$\mathsf{S} \models \varphi \Rightarrow \vdash_{\Lambda} \varphi.$$

It has been proved **S4** is both sound and complete with respect to the class of all topological spaces.[3] The proof uses the following facts:

- **S4** is sound and complete with respect to the class of all reflexive transitive frames.
- A topology can be put on any reflexive transitive frame such that the truth of the formulas is preserved on the resultant topo-model.

[3] Aiello, Pratt-Hartmann, van Bentham: Handbook of Spatial Logics (2007)

Another Proof of the Topo-completeness of **S4**

We construct a special topo-model which we will use to prove the completeness of **S4** with respect to the class of all topological spaces.

The construction is similar to the construction of the canonical model in the case of Kripke semantics [4].

But before that we need a few definitions.

[4] Blackburn, P., de Rijke, M. and Venema, Y.: Modal Logic (2001)

Consistency

Let Λ be a normal modal logic.

Definition

A set of formulas Γ is said to be Λ-consistent if for no finite set $\{\phi_1, \ldots, \phi_n\} \subseteq \Gamma$, we have

$$\vdash_\Lambda (\phi_1 \wedge \ldots \wedge \phi_n) \to \bot,$$

and a Λ-consistent set of formulas Γ is called Λ-maximally consistent (or MCSs) if there is no consistent set of formulas properly containing Γ.

MCSs are indeed maximal

MCSs behave nicely with respect to the underlying logic in the following way.[5]

Lemma

If Γ is an MCS of formulas for the normal logic Λ, then we have

- Γ *is closed under modus pones: if* $\phi, \phi \rightarrow \psi \in \Gamma$, *then* $\psi \in \Gamma$,
- $\Lambda \subseteq \Gamma$,
- *for all formulas* $\phi : \phi \in \Gamma$ *or* $\phi \in \Gamma$,
- *for all formulas* $\phi, \psi : \phi \vee \psi \in \Gamma$ *iff* $\phi \in \Gamma$ *or* $\psi \in \Gamma$.

[5] Blackburn, P., de Rijke, M. and Venema, Y.: Modal Logic (2001)

Are MCSs even there?

Lemma (Lindenbaum's Lemma)

If Σ is a Λ-consistent set of formulas, then there is an MCS Σ^+ such that

$$\Sigma \subseteq \Sigma^+.$$

The proof is constructive[6]

The key step is to enumerate the set of all modal formulas, and keep adding ϕ or $\neg\phi$ to Σ such that the consistency of Σ is preserved.

[6] Blackburn, P., de Rijke, M. and Venema, Y.: Modal Logic (2001)

The Canonical Topological Space

From now on we fix our logic to be **S4**.

We now build a topo-model, using maximally consistent sets, which is complete with respect to **S4**.

Definition (Canonical Topological Space)

It is the pair $\mathcal{X} = \langle X^{\mathcal{L}}, \tau^{\mathcal{L}} \rangle$ where:

- $X^{\mathcal{L}}$ is the set of all maximally consistent sets.
- For each formula ϕ,

$$\widehat{\phi} := \{x \in X^{\mathcal{L}} \mid \phi \in x\},$$

$$B^{\mathcal{L}} = \{\widehat{\Box\phi} \mid \phi \text{ is a formula}\},$$

  and $\tau^{\mathcal{L}}$ is the topology generated by taking $B^{\mathcal{L}}$ as the basis.

The Canonical Topological Space (Cont'd)

Lemma

$B^{\mathcal{L}}$ forms a basis for the set $\mathcal{X}^{\mathcal{L}}$.

What needs to be shown:

1. For each $x \in X^{\mathcal{L}}$, there is a corresponding formula ϕ_x such that $x \in \widehat{\Box\phi_x}$.
2. If $x \in \widehat{\Box\phi} \cap \widehat{\Box\psi}$, then there is a formula χ such that $x \in \widehat{\Box\chi} \subseteq \widehat{\Box\phi} \cap \widehat{\Box\psi}$.

The First Condition

$\top$ is a propositional tautology, so

$$\top \in \mathbf{S4}.$$

By necessitation,

$$\Box\top \in \mathbf{S4}.$$

For each $x \in \mathcal{X}^{\mathcal{L}}$,

$$\mathbf{S4} \subseteq x,$$

so

$$\Box\top \in x,$$

which implies

$$x \in \widehat{\Box\top}.$$

The Second Condition

It can be shown that for arbitrary formulas ϕ and ψ, we have

$$\widehat{\Box\phi} \cap \widehat{\Box\psi} = \Box\widehat{(\phi \wedge \psi)}.$$

This is proved by showing that

$$\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$$

and

$$\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$$

are in **S4**. Then as each $x \in \mathcal{X}^{\mathcal{L}}$ is closed under modus ponens, the claim follows. Once this is shown, $\phi \wedge \psi$ is a candidate for χ.

The Canonical Topo-model

We select an apporpriate valuation for the canonical topological space.

Definition (Canonical Topo-model)

The canoncial topo-model is the pair $M^{\mathcal{L}} = \langle \mathcal{X}, v^{\mathcal{L}} \rangle$, where

- $\mathcal{X}$ is the canonical topological space,
- $v^{\mathcal{L}}(p) = \{x \in X^{\mathcal{L}} \mid p \in x\}$.

Lemma (Truth Lemma)

For all modal formulas ϕ, for all $x \in X^{\mathcal{L}}$,

$$M^{\mathcal{L}}, x \models \phi \Leftrightarrow x \in \widehat{\phi}.$$

The proof of the truth lemma is done by the induction on the set of all formulas.

Completeness

Corollary (Completeness)

S4 *is complete with respect to the class of all topological spaces.*

Key steps (contrapositive):

- Let
$$\phi \notin \mathbf{S4}.$$
Then $\{\neg\phi\}$ is consistent.
- $\{\neg\phi\}$ can be extended to a maximally consistent set by the *Lindenbaum Lemma*. Let it be Σ.
- By the truth lemma we have
$$M^{\mathcal{L}}, \Sigma \models \neg\phi$$
which implies
$$M^{\mathcal{L}}, \Sigma \not\models \phi.$$

Preliminaries to the McKinsey-Tarski Theorem

The McKinsey-Tarski Theorem

The McKinsey-Tarski theorem is stronger than what we have proved till now.

It says that **S4** is sound and complete with respect to the class of dense-in-itself seperable metric spaces.

Why stronger?

Every formula not in **S4** is falsifiable on a dense-in-itself seperable metric space.

Topo-bisimulations[7]

Definition (Topo-bisimulation)

A topological bisimulation or simply a topo-bisimulation between two topo-models $M = \langle X, \tau, v \rangle$ & $M' = \langle X', \tau', v' \rangle$ is a nonempty relation $T \subseteq X \times X'$ such that if xTx' then:

1. (atomic clause) for each $p \in P$, $x \in v(p)$ iff $x' \in v'(p)$.
2. (forth) for arbitrary $U \in \tau$, $x \in U$ implies there exists some $U' \in \tau'$ with $x' \in U'$ such that for each $y' \in U'$ there exists some $y \in U$ with yTy'.
3. (back) for arbitrary $U' \in \tau'$, $x' \in U'$ implies there exists some $U \in \tau$ with $x \in U$ such that for each $y \in U$ there exists some $y' \in U'$ with yTy'.

[7] Aiello, Pratt-Hartmann, van Bentham: Handbook of Spatial Logics (2007).

Restating the Conditions

We give a formulation of all the three conditions which is point independent and wholly in terms of open sets and the valuations.

Let R be a relation, with $R \subseteq X \times X'$. Then, for any $A \subseteq X$, and $A' \subseteq X'$, we define

$$R(A) = \{x' \in X' \mid xRx' \text{ for some } x \in A\},$$

and

$$R^{-1}(A') = \{x \in X \mid xRx' \text{ for some } x' \in A'\}.$$

Restating the Conditions (Cont'd)

Lemma (Restating the conditions)

A nonempty relation $T \subseteq X \times X'$ is a topo-bisimulation between two topo-models $M = \langle X, \tau, v \rangle$ & $M' = \langle X', \tau', v' \rangle$ iff

- *(atomic clause) for each propositional letter p,*

$$T(v(p)) \subseteq v'(p) \;\&\; T^{-1}(v'(p)) \subseteq v(p),$$

- *(forth) for each $U \in \tau$, $T(U)$ is open, and*
- *(back) for each $U' \in \tau'$, $T^{-1}(U')$ is open.*

We will be using the above criterion to prove the completeness of **S4** with respect to $\mathbb{Q}$ with the euclidean topology.

An Invariance Result

The concept of a topo-bisimulation is similar to the concept of a bisimulation for the Kripke semantics of the basic modal language[8].

Topo-bisimulations preserve truth formulas on related points.

Theorem

Let $M = \langle X, \tau, v \rangle$ and $M' = \langle X', \tau', v' \rangle$ be two topo-models and $x \in X$, $x' \in X'$ be two topo-bisimilar points. Then for each modal formula ϕ, we have

$$M, x \vDash \phi \text{ iff } M', x' \vDash \phi.$$

That is, modal formulas are invariant under topo-bisimulations.

The proof is by induction on the set of all formulas.

[8] Blackburn, de Rijke, and Venema: Modal Logic (2001).

Linear Orders

Definition (Dense linear order with no endpoints)

Let X be a set and $<$ be a binary relation on X. Then $(X, <)$ is said to be a *linear order* if it follows the following properties:

- (Transitivity) for all $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$, and
- (Trichotomy) for all $x, y \in X$, exactly one of $x < y$, $y < x$ and $x = y$ hold.

A linear order is said to be *dense* if for all $x, y \in X$, if $x < y$, then there exists $z \in X$ such that $x < z < y$. A linear order is said to have *no endpoints* if for all $x \in X$, there exist $y, z \in X$ such that $y < x < z$.

Example

$(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ where $<$ is the usual 'less than' are a dense linearly ordered sets with no endpoints. $(\mathbb{N}, <)$ where $<$ is the usual 'less than' is a linearly ordered set, but is neither dense nor has no endpoints.

Order Isomorphism

Definition (Order isomorphism)

Let $(A, <_A)$ and $(B, <_B)$ be two linearly ordered sets. A function $f : A \to B$ is said to be an *order isomorphism* if

- f is bijective.
- for each $a_1, a_2 \in A$, we have

$$a_1 <_A a_2 \Leftrightarrow f(a_1) <_B f(a_2).$$

Example

Any two finite linearly ordered sets with the same number of elements are order isomorphic.

Cantor's Theorem

Theorem (Cantor)

Any two countable dense linearly ordered sets with no endpoints are isomorphic [9].

The proof uses a method called the *back and forth* method.

The idea is to make an 'increasing' sequence of partial functions which preserve order, such that their union is bijective and order preserving.

[9] Kuratowski, K. and Mostowski, A.: Set Theory (1976).

Homeomorphism between Linearly Ordered Sets

Definition

For a linearly ordered set $(X, <)$, for each $a, b \in X$, we define

$$(a, b) = \{x \in X \mid a < x < b\}.$$

Lemma

If $(X, <)$ is a linearly ordered set with no endpoints then $\{(a, b) \mid a, b \in X\}$ forms a basis.

Homeomorphism between Linearly Ordered Sets (Cont'd)

Definition (Order topology)

We call the topology generated by the above basis as the *order topology* on X.

Example

For the euclidean topology on $\mathbb{Q}$, a basis is the set of open intervals, which also forms a basis for the order topology on $(\mathbb{Q}, <)$. Thus, the order topology and the euclidean topology on $\mathbb{Q}$ are same.

Theorem

Every countable dense linearly ordered set with no endpoints is homeomorphic to $\mathbb{Q}$ with the euclidean topology.

The proof uses the previously mentioned Cantor's theorem, and the order isomorphism itself acts as the homeomorphism.

Interior Maps

Let $\langle X, \tau \rangle$ and $\langle X', \tau' \rangle$ be topological spaces.

Definition (Interior Map)

A function $f : X \to X'$ is said be an interior map if

- f is continuous, and
- f is open.

Basically, f is an interior map if

- $f(U)$ is open for each $U \in \tau$.
- $f^{-1}(U')$ is open for each $U' \in \tau'$.

(This should remind of the back and forth condition of topo-bisimulations.)

Frames and Alexandroff topology

Frame: A set X equipped with a binary relation R.

Reflexive/Transitive Frame: R is a reflexive/transitive relation.

S4 Frame: Reflexive transitive frame.

Definition (Upsets)

Let (X, R) be an **S4**-frame. A subset A of X is called an upset if for each $x, y \in X$, if $x \in A$ and Rxy holds, then $y \in A$.

Upsets are subsets which are closed with respect to the relation R.

Upsets: Example

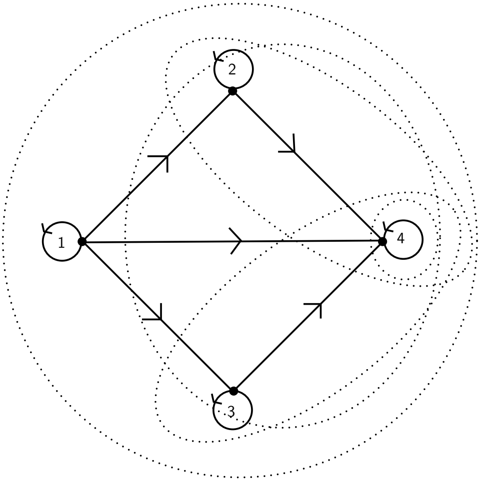

Figure: All the upsets (except $\emptyset$) of an **S4**-frame

Completeness of **S4**

Proposition

Let (X, R) be an **S4**-frame. Then, for

$$\tau_R = \{ A \subseteq X \mid A \text{ is an upset}\},$$

(X, τ_R) forms a topological space.

Lemma

Let $\mathfrak{M} = (X, R, v)$ be a model based on an **S4**-frame. Let M be the topomodel (X, τ_R, v). Then for all modal formulas φ and all $x \in X$ we have

$$\mathfrak{M}, x \vDash \varphi \text{ iff } M, x \vDash \varphi.$$

The McKinsey-Tarski Theorem

The Plan

Frame: A set X equipped with a binary relation R.

Rooted: For the frame (X, R), there is an $r \in X$ such that for each $w \in X$, rRw.

Reflexive/Transitive Frame: R is a reflexive/transitive relation.

The logic **S4** is complete with respect to the class of all finite rooted reflexive transitive frame.

(Every formula not in **S4** can be falsified on a model based on a finite rooted reflexive transitive frame. This is done by falsifying the formula on the canonical relational model and then filtering the model and choosing one of its generated submodel appropriately[10].)

[10] Blackburn, de Rijke, Venema: Modal Logic (2001)

The Plan (Cont'd)

We construct a countably finite linearly ordered dense set $(\Sigma, <)$ with no endpoints.

Let μ represent the order topology on Σ. Then by Cantor's theorem, (Σ, μ) is homeomorphic to $\mathbb{Q}$ with the euclidean topology.

Next, we construct an interior map from from Σ to an arbitrary finite rooted **S4** frame where the topology on the frame is the Alexandroff topology.

Thus, we get an interior map from the $\mathbb{Q}$ to the frame.

Using the interior map, we construct a topo-bisimulation. As the truth of formulas is invariant under topo-bisimulations, a formula not in **S4** is falsifiable on a topo-model based on $\mathbb{Q}$, which is a dense-in-itself seperable metric space.

The Set Σ

Let

$$\Sigma = \{\text{finite sequences on non-zero integers}\}.$$

Let Λ denote the empty sequence. Then Σ contains Λ. Some other elements of Σ are $1, -343, 211, 2$, and $2139 - 87 - 3 - 981 - 26$.

We want to put an order on Σ such that it becomes a countable dense linearly ordered set with no endpoints.

The ordered set $(\Sigma, <)$.

It has been proved that[11] such an order $<$ can be defined such that

- Σ is countably infinite,
- $(\Sigma, <)$ is linearly ordered,
- $(\Sigma, <)$ is dense,
- $(\Sigma, <)$ has no-endpoints.

Let μ denote the order topology on Σ. Then, by the Cantor's theorem (Σ, μ) is homeomorphic to $\mathbb{Q}$ with the euclidean topology.

[11] Lucero-Bryan: The d-Logic of the Rational Numbers: A Fruitful Construction (2011).

The Interior Map

Let (X, R) be a finite rooted **S4**-frame.

Then, there exists an onto function f from Σ to X such that f is an interior map[12].

Let the homeomorphism from $\mathbb{Q}$ to Σ be g.

Then, $f \circ g$ is an interior map from $\mathbb{Q}$ to X.

[12] Lucero-Bryan: The d-Logic of the Rational Numbers: A Fruitful Construction (2011).

From Interior Map to Topo-bisimulation

Given a valuation v_X on X, we define a corresponding valuation $v_{\mathbb{Q}}$ on the topological space $\mathbb{Q}$ as following: For each propositional letter p,

$$v_{\mathbb{Q}}(p) := (f \circ g)^{-1}(v_X(p)).$$

Let

$$T := \{(q, x) \mid x = f \circ g(q)\}.$$

Then, by the virtue of $f \circ g$ being an interior map, we get that T is a topo-bisimulation.

Putting it all together

Let $\phi \notin$ **S4**.

Then, there is a Kripke model based on a finite rooted reflexive transitive frame where it is falsifiable.

As there is a topo-model based on $\mathbb{Q}$ with the euclidean topology, which is topo-bisimilar to the Kripke model, so ϕ is falsifiable on the topo-model.

Hence, **S4** is complete with respect to the class of all dense-in-itself seperable metric space.

Thank You

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